arXiv:hep-th/0011172v2 9 Jan 2001

Quantum Mechanics on the Noncommutative Plane and Sphere

V. P. NAIR^a and A.P. POLYCHRONAKOS^{b,} [1]

^{a,b}*Physics Department, City College of the CUNY*
New York, NY 10031

^{a,b}*The Graduate School and University Center*
City University of New York
New York, NY 10016

^{b}*Physics Department, Rockefeller University*
New York, NY 10021

E-mail: vpn@ajanta.sci.ccny.cuny.edu, poly@teorfys.uu.se

Abstract

We consider the quantum mechanics of a particle on a noncommutative plane. The case of a charged particle in a magnetic field (the Landau problem) with a harmonic oscillator potential is solved. There is a critical point with the density of states becoming infinite for the value of the magnetic field equal to the inverse of the noncommutativity parameter. The Landau problem on the noncommutative two-sphere is also solved and compared to the plane problem.

---

[1]On leave from Theoretical Physics Dept., Uppsala University, Sweden and Physics Dept., University of Ioannina, Greece.

1 Introduction

Noncommutative spaces can arise as brane configurations in string theory and in the matrix model of M-theory [1]. Fluctuations of branes are described by gauge theories and thus, motivated by the existence of noncommutative branes, there has recently been a large number of papers dealing with gauge theories, and more generally field theories, on such spaces [2]. However, there has been relatively little work exploring the quantum mechanics of particles on noncommutative spaces. Since the one-particle sector of field theories, which can be treated in a more or less self-contained way in the free field or weakly coupled limit, leads to quantum mechanics, the brane connection suggests that a more detailed study of this topic should be useful. This is the subject of the present paper.

Some of the algebraic aspects of quantum mechanics on spaces with an underlying Lie algebra structure were considered in reference [3]. The noncommutative plane can be defined in terms of a projection to the lowest Landau level of dynamics on the commuting plane [4]; some features of particle dynamics in terms of a similar construction were contained in [5]. The spectrum of a harmonic oscillator on the noncommutative plane was derived in [6] and the case of a general central potential was recently discussed in [7]. In this paper, we will analyze the algebraic structures in more detail. We will solve the problem of a charged particle in a magnetic field (the Landau problem) with an oscillator potential on the noncommutative plane. There is an interesting interplay of the magnetic field B and the noncommutativity parameter θ, with a critical point at $B\theta = 1$ where the density of states becomes infinite. We also solve the Landau problem on the noncommutative sphere, for which the basic algebraic structure turns out to be $SU(2) \times SU(2)$. We also show how the results on the plane can be recovered in the limit of a large radius for the sphere.

2 The noncommutative plane

We start with the quantum mechanics of a particle on the noncommutative two-dimensional plane. For single particle quantum mechanics, we need the Heisenberg algebra for the position and momentum operators. The two-dimensional noncommutative plane is described by the coordinates x_1, x_2 which obey the commutator algebra $[x_1, x_2] = i\theta$ where θ is the noncommutativity parameter. With the momentum operator p_i, $i = 1, 2$, we may write the full Heisenberg algebra as

$$\begin{aligned} [x_1, x_2] &= i\theta \\ [x_i, p_j] &= i\delta_{ij} \\ [p_1, p_2] &= 0 \end{aligned} \qquad (1)$$

The fact that x_1 and x_2 commute to a constant may suggest that they can themselves serve as translation operators. However, this is not adequate to obtain the last of the relations (1); one needs independent operators. A realization of the momentum operators, for example,

would be

$$p_1 = \frac{1}{\theta}\left(x_2 + k_1\right)$$
$$p_2 = \frac{1}{\theta}\left(-x_1 + k_2\right) \tag{2}$$

with $[k_1, k_2] = -i\theta$ and $[k_i, x_j] = 0$. In this case, (x_1, x_2) and (k_2, k_1) obey identical commutation rules and are mutually commuting. p_i are thus constructed from two copies of the x-algebra.

We may use the realization (2) of the p_i to solve specific quantum mechanical problems. However, before turning to specific examples, some comments about the p_i-operators are in order. In the usual quantum mechanics with commuting x's, a single irreducible representation for the x-algebra would be given by $x_i = c_i$ for fixed real numbers c_i. Coordinate space is spanned by an infinity of irreducible representations of the x-algebra. Additional independent operators p_i are needed to obtain a single irreducible representation, now for the augmented set of operators. The p_i's connect different irreducible representations of the x-algebra. In order to recover this structure for small θ, we need the independent set of operators k_i in (2).

Single particle quantum mechanics may also be viewed as the one-particle sector of quantum field theory, in the free field or very weakly coupled limit, with the Schrödinger wave function obeying essentially the free field equation. Since quantum field theories on non-commutative manifolds have already been defined and investigated to some extent, this may seem to give a quick and simple way to write down one-particle quantum mechanics. The case of a nonrelativistic Schrödinger field suffices to illustrate the point. The field $\Phi(x)$ is a function of the noncommuting coordinates x_i. The action for this field in an external potential and coupled to a gauge field may be written as

$$\mathcal{S} = \int dt \, \mathrm{Tr}\left[\Phi^\dagger i D_0 \Phi - \frac{(D_i\Phi)^\dagger (D_i\Phi)}{2m} - \Phi^\dagger V \Phi\right] \tag{3}$$

where $D_\mu\Phi = \partial_\mu\Phi + \Phi A_\mu$. Even though we have indicated the derivative as $\partial_\mu\Phi$, it should be emphasized that, since Φ is noncommutative, even classically, translations must be implemented by taking commutators with an operator conjugate to x. This is implicit in the definition of $\partial_\mu\Phi$. Further, in (3), the gauge fields act on the right of the field Φ and the potential on the left. This ensures that the action of the gauge field and the potential commute and allows an unambiguous separation of these two types of interaction terms. The one-particle wavefunction is the matrix element of Φ between the vacuum and one-particle states. The equation of motion for (3) is

$$iD_0\Phi + D_i(D_i\Phi) - V\Phi = 0 \tag{4}$$

and taking the appropriate matrix element, we see that the Schrödinger equation has a similar form, with the qualification that the action of derivatives is defined via commutators with an operator conjugate to x.

3

With the algebra of the x_i's in (1), we can see that translations of the argument of Φ may be achieved using just the x_i's themselves by writing [8]

$$- i\partial_i\Phi = \frac{1}{\theta} \ [\epsilon_{ij}x_j, \Phi] = \frac{\epsilon_{ij}x_j}{\theta} \ \Phi - \Phi \ \frac{\epsilon_{ij}x_j}{\theta} \tag{5}$$

In other words, by using the adjoint action of $\epsilon_{ij}x_j$, we can obtain translations on functions of x_i. It is then easy to check that $[\partial_1, \partial_2] = 0$. Translations thus involve the left and right actions of the x_i's on Φ, which are mutually commuting actions. Since we do not usually take commutators of operators with the wavefunction in quantum mechanics, it is preferable, in going to the one-particle case, to replace the right action of the x's on Φ formally by a left action by

$$x_{iR}\Phi = \Phi x_i \tag{6}$$

This can be done in more detail as follows. If we realize the x-algebra on a Hilbert space $\mathcal{H}$, then Φ is an element of $\mathcal{H} \otimes \tilde{\mathcal{H}}$, where $\tilde{\mathcal{H}}$ is the dual Hilbert space and we can write $\Phi = \sum_{mn} \Phi_{mn}|m\rangle\langle n|$ in terms of a basis $\{|m\rangle\}$. Mapping the elements of $\tilde{\mathcal{H}}$ to the corresponding elements of $\mathcal{H}$ in the standard way, we can introduce $\Phi' = \sum_{mn} \Phi_{mn}|m\rangle \otimes |n\rangle$. The right action of x's on Φ is then mapped onto the left action on Φ' as given above. Note that $[x_{iR}, x_{jR}]\Phi = -\Phi[x_i, x_j] = -i\theta\epsilon_{ij}\Phi$. We can thus identify $-\epsilon_{ij}x_{jR}$ as k_i and we obtain the realization given in (2). The one-particle limit of field theory thus naturally leads to the structure of two mutually commuting copies of the x-algebra.

We see that from both points of view, namely, of one-particle quantum mechanics as defined by an irreducible representation of the Heisenberg algebra generalized to include noncommutativity of coordinates, or as defined by the one-particle limit of field theory, we are led to the algebraic structure (1, 2). This result is consistent with the discussion of quantum mechanics on the noncommuting two-sphere given in reference [3]. In that case also, one had two mutually commuting copies of the x-algebra, which was $SU(2)$. The momentum operator was then constructed from the $SU(2) \times SU(2)$ algebra in a way analogous to the realization (2). The present results for the plane may in fact be obtained, as we shall see later, for a small neighborhood of the sphere, in the limit of large radius.

A concrete and simple example which illustrates the general discussion so far is provided by the harmonic oscillator on the noncommutative plane. It is not any more difficult to solve the more general case of a charged particle in a magnetic field (the Landau problem) with a quadratic (or oscillator) potential and so we shall treat this case below. The fact that we have a magnetic field B can be incorporated by modifying the commutation rule for the momenta to $[p_1, p_2] = iB$. In other words, B measures the noncommutativity of the momenta. The interplay of B and θ can thus lead to some interesting behavior. Denoting the position and momentum operators by ξ_i, $i = 1, ..., 4$, $\xi = (x_1, x_2, p_1, p_2)$, the commutation rules are

$$
\begin{aligned}
[\xi_i, \xi_j] \quad &= iP_{ij} \\
P \quad &= \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & B \\ 0 & -1 & -B & 0 \end{pmatrix}
\end{aligned} \tag{7}
$$

4

The Hamiltonian for the oscillator with magnetic field is

$$H = \frac{1}{2} \left[p_1^2 + p_2^2 + \omega^2 (x_1^2 + x_2^2) \right] \tag{8}$$

It is obviously invariant under rotations in the plane. The angular momentum, being the generator of these rotations, takes the form

$$L = \frac{1}{1 - \theta B} \left[x_1 p_2 - x_2 p_1 + \frac{B}{2} (x_1^2 + x_2^2) + \frac{\theta}{2} (p_1^2 + p_2^2) \right] \tag{9}$$

We observe that it acquires θ-dependent corrections compared to the commutative case.

The algebra (7) has many possible realizations. The 'minimal' one in terms of two independent sets of canonical coordinates and momenta $(\bar{x}_i, \bar{p}_i)$ satisfying standard Heisenberg commutation relations would be

$$\begin{aligned} x_1 &= \bar{x}_1, & p_1 &= \bar{p}_1 + B\bar{x}_2 \\ x_2 &= \bar{x}_2 + \theta \bar{p}_1, & p_2 &= \bar{p}_2 \end{aligned} \tag{10}$$

We prefer, however, to use a realization as close to (2) as possible to maintain contact with noncommutative field theory. Using the realization (2) for the momenta, we find $[k_1, k_2] = i(B - (1/\theta))$. Because of this, the cases $B < 1/\theta$ and $B > 1/\theta$ should be treated differently. Consider first the case $B < 1/\theta$. In this case, we can define

$$\begin{aligned} x_1 &= l\alpha_1, & p_1 &= \frac{1}{l}\beta_1 + q\alpha_2 \\ x_2 &= l\beta_1, & p_2 &= \frac{1}{l}\alpha_1 - q\beta_2 \end{aligned} \tag{11}$$

where $l^2 = \theta$ and $q^2 = (1/\theta) - B$. α_i, β_i form a set of canonical variables, i.e., $[\alpha_i, \beta_j] = i\delta_{ij}$. The Hamiltonian for the oscillator with the magnetic field is given by

$$\begin{aligned} H &= \frac{1}{2} \left[p_1^2 + p_2^2 + \omega^2 (x_1^2 + x_2^2) \right] \\ &= \frac{1}{2} \left[\left(\omega^2 l^2 + \frac{1}{l^2} \right) (\alpha_1^2 + \beta_1^2) + q^2 (\alpha_2^2 + \beta_2^2) + \frac{2q}{l} (\alpha_1 \beta_2 + \alpha_2 \beta_1) \right] \end{aligned} \tag{12}$$

We can now make a Bogolyubov transformation on this by expressing α_i, β_i in terms of a canonical set Q_i, P_i by writing

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \cosh \lambda \begin{pmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{pmatrix} + \sinh \lambda \begin{pmatrix} P_2 \\ P_1 \\ Q_2 \\ Q_1 \end{pmatrix} \tag{13}$$

Choosing

$$\tanh 2\lambda = - \frac{2ql}{1 + \omega^2 l^4 + q^2 l^2} \tag{14}$$

5

the Hamiltonian (12) becomes

$$H = \frac{1}{2}\left[\Omega_+ \left(P_1^2 + Q_1^2\right) + \Omega_-(P_2^2 + Q_2^2)\right] \tag{15}$$

where

$$\Omega_\pm = \tfrac{1}{2}\sqrt{(\omega^2\theta - B)^2 + 4\omega^2} \ \pm \tfrac{1}{2}(\omega^2\theta + B) \tag{16}$$

Equation (15) shows that the spectrum is given by that of two harmonic oscillators of frequencies Ω_+ and Ω_-.

The case of $B > 1/\theta$ can be treated in a similar way. With $q^2 = B - (1/\theta)$, we can write

$$x_1 = l\alpha_1, \qquad\qquad p_1 = \frac{1}{l}\beta_1 + q\alpha_2$$

$$x_2 = l\beta_1, \qquad\qquad p_2 = -\frac{1}{l}\alpha_1 + q\beta_2 \tag{17}$$

In terms of the α_i, β_i, the Hamiltonian becomes

$$H = \frac{1}{2}\left[\left(\omega^2 l^2 + \frac{1}{l^2}\right)(\alpha_1^2 + \beta_1^2) + q^2(\alpha_2^2 + \beta_2^2) + \frac{2q}{l}(\alpha_2\beta_1 - \alpha_1\beta_2)\right] \tag{18}$$

The required Bogolyubov transformation is

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \cos\lambda \begin{pmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{pmatrix} + \sin\lambda \begin{pmatrix} P_2 \\ P_1 \\ -Q_2 \\ -Q_1 \end{pmatrix} \tag{19}$$

The required choice of λ is given by

$$\tan 2\lambda = \frac{2ql}{1 + \omega^2 l^4 - q^2 l^2} \tag{20}$$

H can then be written as in (15) with

$$\Omega_\pm = \pm\tfrac{1}{2}\sqrt{(\omega^2\theta - B)^2 + 4\omega^2} \ + \tfrac{1}{2}(\omega^2\theta + B) \tag{21}$$

We again have two oscillators of frequencies $\Omega_\pm$.

We see from the above results that there is a critical value of the magnetic field or θ given by $B\theta = 1$. Ω_- vanishes upon approaching this value from either side. The Hamiltonian is independent of P_2, Q_2. Thus the number of states for fixed energy will become unbounded, since all the states generated by P_2, Q_2 are now degenerate. This large degeneracy can also be seen from a semiclassical estimate of the number of states for fixed energy. Going back to (7), we see that $\det P = (1 - B\theta)^2$. The phase volume is thus given by

$$\begin{aligned} d\mu \ &= \ \frac{1}{\sqrt{\det P}} \ dx_1 dx_2 dp_1 dp_2 \\ &= \ \frac{1}{|1 - \theta B|} \ dx_1 dx_2 dp_1 dp_2 \end{aligned} \tag{22}$$

6

Surfaces of equal energy in phase space are ellipsoids defined by $E = \frac{1}{2}(p_1^2 + p_2^2 + \omega^2 x_1^2 + \omega^2 x_2^2)$. A semiclassical estimate of the number of states with energy less that E is given by the volume inside this surface divided by $(2\pi)^2$. We obtain

$$N = \frac{V}{(2\pi)^2} = \frac{1}{2|1 - B\theta|}\left(\frac{E}{\omega}\right)^2 \tag{23}$$

The criticality of the point $\theta B = 1$ is once again clear; the density of states is infinite at this point.

When $\omega^2 = 0$ we have the pure Landau problem. In this case $\Omega_+ = B$, $\Omega_- = 0$ for $B > 0$, or $\Omega_+ = 0$, $\Omega_- = |B|$ for $B < 0$ and we have the standard, infinitely degenerate Landau levels as in the commutative case. The density of states per unit area, denoted by ρ, however, is now modified to

$$\rho = \frac{1}{2\pi}\left|\frac{B}{1 - \theta B}\right| \tag{24}$$

To demonstrate this, observe that the magnetic translations, defined as the operators performing translations on x_i and commuting with the Hamiltonian, are now

$$D_1 = \frac{1}{1 - \theta B}(p_1 - Bx_2), \quad D_2 = \frac{1}{1 - \theta B}(p_2 + Bx_1) \tag{25}$$

These are the operators responsible for the infinite degeneracy of the Landau levels and their commutator determines the density of degenerate states on the plane. D_i commute with x_j in the standard way, $[x_i, D_j] = i\delta_{ij}$, but their mutual commutator is now

$$[D_1, D_2] = -i\frac{B}{1 - \theta B} \tag{26}$$

which reproduces the result (24). We observe that for the critical value of the magnetic field $B = 1/\theta$ the density of states on the plane becomes infinite. The same result can also be obtained in the semiclassical way of the previous paragraph, where now we calculate the phase space volume of a circle $E = \frac{1}{2}(p_1^2 + p_2^2)$ in momentum space times a domain of area A in coordinate space. The result is

$$N = \frac{V}{(2\pi)^2} = \frac{EA}{2\pi|1 - \theta B|} \tag{27}$$

which is compatible with (24) upon filling the lowest n Landau levels such as $E = n|B|$.

It is also interesting to calculate the magnetic length in this case, that is, the minimum spatial extent of a wavefunction in the lowest Landau level. This can be achieved by putting both oscillators Ω_+ and Ω_- in their ground state: the one with nonvanishing frequency excites Landau levels while the one with vanishing frequency creates annular states on the plane for each Landau level. In this state we have $\langle P_i^2 \rangle = \langle Q_i^2 \rangle = \frac{1}{2}$ and $\langle P_i \rangle = \langle Q_i \rangle = 0$. Using (11), (13) and (14) we can calculate $\langle x_i^2 \rangle$ for $B < 1/\theta$ as

$$\langle x_1^2 + x_2^2 \rangle = l^2(\cosh^2 \lambda + \sinh^2 \lambda) = \frac{1}{|B|}(2 - B\theta) \tag{28}$$

7

while for $B > 1/\theta$ we obtain from (17) and (19)

$$\langle x_1^2 + x_2^2 \rangle = l^2(\cos^2 \lambda + \sin^2 \lambda) = \theta \tag{29}$$

So we see that for subcritical magnetic field the magnetic length is more or less as in the commutative case while for overcritical one it assumes the value $l = \sqrt{\theta}$ which is the minimal uncertainty on the noncommutative plane.

We conclude by noting that the oscillator frequency ω and magnetic field B appearing in the Hamiltonian are distinct from the corresponding 'kinematical' quantities that appear in the equations of motion. Expressing the equations of motion in terms of x_i and its time derivatives we obtain

$$\ddot{x}_i = (B + \theta\omega^2)\epsilon_{ij}\dot{x}_i - (1 - \theta B)\omega^2 x_i \tag{30}$$

We recognize the Lorenz force and the spring force with effective magnetic field and spring constant

$$\tilde{B} = B + \theta\omega^2, \quad \tilde{\omega}^2 = (1 - \theta B)\omega^2 \tag{31}$$

The spectral frequencies $\Omega_\pm$ in terms of the kinematical parameters become identical to the corresponding noncommutative ones, namely

$$\Omega_\pm = \left| \pm\tfrac{1}{2}\sqrt{\tilde{B}^2 + 4\tilde{\omega}^2} + \tfrac{1}{2}\tilde{B} \right| \tag{32}$$

In this parametrization the noncommutativity of space manifests only through the density of states and spatial correlation functions. Interestingly, for the critical value $B = 1/\theta$ the oscillator ω transmutates entirely into a magnetic field $\tilde{B} = \theta\omega^2 + \theta^{-1}$

3   The noncommutative sphere

We now turn to the quantum mechanics of a particle on the noncommutative two-sphere. To set the stage, we first give a review of the commutative sphere with a magnetic monopole at the center. The observables of the theory consist of the particle coordinates x_i and the angular momentum generators J_i, $i = 1, 2, 3$. Their algebra is

$$\begin{aligned}
[x_i, x_j] &= 0 \\
[J_i, x_j] &= i\epsilon_{ijk}x_k \\
[J_i, J_j] &= i\epsilon_{ijk}J_k
\end{aligned} \tag{33}$$

while the Hamiltonian is taken to be

$$H = \frac{1}{2x^2}J^2 \tag{34}$$

The algebra (33) has two Casimirs, which can be chosen to have fixed values, say,

$$\begin{aligned}
x^2 &= a^2 \\
x \cdot J &= -\frac{n}{2}a
\end{aligned} \tag{35}$$

The first one is simply the square of the radius of the sphere. In the second one, n can be identified as the monopole number. Indeed, in the presence of a monopole field the angular momentum acquires a term in the radial direction proportional to the monopole number which makes the second Casimir nonvanishing. The interpretation as a magnetic field can be independently justified by deriving the equations of motion of x_i using the Hamiltonian (34)

$$\ddot{x}_i = -\frac{1}{2}\left[\left(\frac{J^2 - (n/2)^2}{a^4}\right)x_i + x_i\left(\frac{J^2 - (n/2)^2}{a^4}\right)\right] + \epsilon_{ijk}\dot{x}_j\frac{nx_k}{2a^3} \tag{36}$$

The first term is a centripetal force, due to the motion on a curved manifold; the kinematical angular momentum squared is seen to be $J^2 - (n/2)^2$. The second term is a Lorentz force, corresponding to a radial magnetic field $B_i = (n/2)\hat{x}_i/a^2$. The monopole number, then, is

$$N = \frac{4\pi a^2 B}{2\pi} = n \tag{37}$$

It is interesting that the magnetic field does not appear as a parameter in the Hamiltonian, not even as a modification of the Poisson structure (as in the planar case), but rather as a Casimir of the algebra of observables.

We now turn to the noncommutative sphere. The quantum mechanics of a particle on a noncommutative sphere was discussed in [3]. The structure of observables is similar, with the difference that the coordinates do not commute but rather form an $SU(2)$ algebra. Specifically,

$$\begin{aligned} [R_i, R_j] &= i\epsilon_{ijk}R_k \\ [J_i, R_j] &= i\epsilon_{ijk}R_k \\ [J_i, J_j] &= i\epsilon_{ijk}J_k \end{aligned} \tag{38}$$

R^2 is a Casimir, as before, but the magnetic Casimir is deformed to $R \cdot J - \frac{1}{2}J^2$.

This operator structure is realized in terms of an $SU(2) \times SU(2)$-algebra with corresponding mutually commuting generators R_i, K_i. In terms of these, the angular momentum is $J_i = R_i + K_i$. We have two Casimir operators, $R^2 = r(r+1)$ and $K^2 = k(k+1)$, and we can label an irreducible representation by the maximal spin values (r, k). The magnetic Casimir becomes $\frac{1}{2}(R^2 - K^2)$. If the radius of the sphere is denoted by a as before, we can identify the coordinates x_i as

$$x_i = \frac{a}{\sqrt{r(r+1)}}R_i \tag{39}$$

The commutative case can be obtained as the limit in which both r and k become very large, but with their difference $k - r = n/2$ being fixed (so that the angular momentum J^2 remain finite). In that limit, the magnetic Casimir becomes

$$\tfrac{1}{2}(R^2 - K^2) = (r - k)\frac{k + r + 1}{2} \simeq -\frac{n}{2}r \tag{40}$$

n becomes the monopole number. We can, therefore, identify the integer $n = 2(k - r)$ as a quantized 'monopole' number in the noncommutative case.

The Hamiltonian of the particle can again be taken proportional to the square of the angular momentum:

$$H = \frac{\gamma}{2a^2} J^2 \tag{41}$$

with γ some coefficient depending on the Casimirs. In the limit of a commutative sphere γ should become 1 in order to reproduce the standard results. In general, however, there is no a priori reason to fix a specific value for γ and, as we shall demonstrate, a different choice must be made in order to recover the limit of the noncommutative plane. The energy spectrum of the particle is clearly

$$E = \frac{\gamma}{2R^2} j(j+1) , \quad j = \frac{|n|}{2}, \frac{|n|}{2} + 1, \ldots j + k \tag{42}$$

Both the energy and angular momentum have a finite spectrum, reflecting the fact that the Hilbert space is finite dimensional.

Comparison to the noncommutative plane can be made by scaling appropriately the parameters of the model. We should take the radius a in (39) to infinity and consider a small neighborhood, say, around the 'north pole' $R_3 = r$, with x_1, x_2 being the relevant coordinates. From the definition (39) of x_i, we then identify the noncommutativity parameter as $\theta \approx a^2/r$. So the scaling of the parameter r is

$$r = \frac{a^2}{\theta} , \quad a \to \infty \tag{43}$$

Only the low-lying states of J^2 and H should be considered in this limit, with $j = \frac{|n|}{2} + l$, $l = 0, 1, 2 \ldots$. Since $R_3 \approx r$ for the states of interest in this limit, we must also have $K_3 \approx -k = -r - n/2$ so that $j \approx |n|/2$. This also means that $J_3 \approx -n/2$. $k = r + n/2$ and γ should then scale appropriately to obtain the planar operator algebra of observables for a particle on the noncommutative plane in the presence of a magnetic field.

The operators $\epsilon_{ij} J_j/a$ $(i, j = 1, 2)$ generate translations of x_i in the planar limit, i.e.,

$$[x_i, \frac{1}{a}\epsilon_{jk}J_k] = \frac{i}{\sqrt{r(r+1)}}\delta_{ij}R_3 \approx i\delta_{ij} \tag{44}$$

So one might be tempted to identify them with the momentum operators in that limit. In the presence of a magnetic field, however, we understand that these should instead become the magnetic translations D_i, since they both commute with the Hamiltonian. Their commutator

$$[D_1, D_2] = \frac{1}{a^2}[J_2, -J_1] = \frac{i}{a^2}J_3 \approx -i\frac{n}{2a^2} \tag{45}$$

should then reproduce the result (26) for the plane. This leads to the identification

$$n = \frac{2Ba^2}{1 - \theta B} \tag{46}$$

which fixes the scaling of n and k. It remains to identify the momenta $p_i = \dot{x}_i$. From the Hamiltonian (41) we obtain

$$\dot{x}_i = \frac{\gamma}{R\sqrt{r(r+1)}}\epsilon_{ijk}K_j R_k \tag{47}$$

10

The commutator of x_i and $p_j = \dot{x}_j$ then becomes

$$[x_i, p_j] = \frac{i\gamma}{r(r+1)} \left(K_i R_j - K_k R_k \delta_{ij} \right) \tag{48}$$

In the planar limit K_3 and R_3 dominate over $K_{1,2}$ and $R_{1,2}$. Therefore, the above commutator becomes, for $i, j = 1, 2$,

$$[x_i, p_j] \approx -\frac{i\gamma}{r^2} K_3 R_3 \delta_{ij} \approx i\gamma \frac{k}{r} \delta_{ij} \tag{49}$$

(we also set $r(r+1) \approx r^2$). To reproduce the canonical commutators on the plane we must set

$$\gamma = \frac{r}{k} = \frac{r}{r + \frac{n}{2}} = 1 - \theta B \tag{50}$$

which fixes the scaling of γ. We can now calculate the commutator of momenta

$$[p_1, p_2] = \frac{i}{k^2 a^2} K \cdot R (K_3 + R_3) \approx i\frac{nr}{2a^2 k} = iB \tag{51}$$

which is, indeed, the correct planar commutator.

Finally, the spectrum of the Hamiltonian becomes

$$E = \frac{\gamma}{2a^2} \left(\frac{|n|}{2} + l \right) \left(\frac{|n|}{2} + l + 1 \right) \approx \frac{\gamma n^2}{8a^2} + \frac{\gamma |n|}{2a^2}(l + \tfrac{1}{2}) = \frac{B^2 a^2}{2(1 - \theta B)} + |B|(l + \tfrac{1}{2}) \tag{52}$$

Apart from a zero-point shift of order a^2, we have agreement with the Landau level spectrum of the noncommutative plane. The above spectrum, but without the zero-point shift, is also reproduced by the low-lying states of the operator $H' = \frac{1}{2}p_i^2$, thus establishing the full correspondence with the plane. The density of states on each Landau level can also be calculated. For a given energy eigenvalue corresponding to $j = |n|/2 + l$ there are $2j + 1$ degenerate states. The space density of these states is

$$\rho = \frac{2j + 1}{4\pi a^2} = \frac{|n|}{4\pi a^2} + \frac{2l + 1}{4\pi a^2} \approx \frac{1}{2\pi} \left| \frac{B}{1 - \theta B} \right| \tag{53}$$

again in agreement with the planar case.

A final word is in order concerning the sphere-plane correspondence. Naively, we may expect that to obtain the full range of magnetic fields $-\infty < B < +\infty$, the integer n should span all values from $-\infty$ to $+\infty$. But clearly this is not possible, since $k = r + n/2 = r(1 + \mu)$ must be nonnegative. In other words, $\mu = n/2r$ must satisfy $\mu > -1$. From (43, 46) we have

$$\mu = \frac{\theta B}{1 - \theta B} \tag{54}$$

We see that the allowed values of μ correspond to $B < 1/\theta$. $B \to 1/\theta$ corresponds to $\mu \to \infty$ and $B \to -\infty$ corresponds to $\mu \to 1$. What about $B > 1/\theta$? For these fields we observe that $\gamma = 1 - \theta B < 0$ and thus the coefficient of J^2 in the Hamiltonian (41) becomes negative. This means that low-lying energy states now correspond to the *highest* value of j (rather

than the lowest), that is, $j = r + k = 2r + n/2$. Since $R_3 \approx r$, we must have $K_3 \approx k$ for such states, and thus $J_3 \approx r + k$. Repeating the previous analysis, we see from (45) that it is $-J_3/R^2$ which is identified with $B/(1 - \theta B)$ and thus

$$r + k = -\frac{BR^2}{1 - \theta B} \quad \text{so that} \quad \mu = \frac{n}{2r} = \frac{2 - \theta B}{\theta B - 1} \tag{55}$$

For $B > 1/\theta$ the above μ spans again the allowed range of values $q > -1$. In summary, the picture is that for each pair of values r, k the energy spectrum of the sphere is bounded between two end levels, one at $j_- = |r - k|$ and one at $j_+ = r + k$. The spectrum around each end level maps to a particular magnetic field in the planar limit, j_- corresponding to subcritical and j_+ to overcritical values of B. Thus the mapping from sphere to plane is actually one to two. This also suggests a duality between magnetic fields B and B' satisfying $B + B' = 2/\theta$, since they both correspond to the same spherical case. The critical field $B = 1/\theta$ is self-dual.

Acknowledgments

References

[1] T. Banks, W. Fischler, S. Shenker and L. Susskind, Phys. Rev. **D55** (1997) 5112; for a recent review, see W. Taylor, Lectures at the NATO School, Iceland, 1999, hep-th/0002016.

[2] There are well over 200 recent papers on this; citations to the following papers will give an overall view of the field. A. Connes, M.R. Douglas and A. Schwarz, hep-th/9711162; N. Seiberg and E. Witten, hep-th/9908142.

[3] V.P. Nair, hep-th/0008027.

[4] G. Dunne, R. Jackiw and C. Trugenberger, Phys. Rev. **D41** (1990) 661.

[5] D. Bigatti and L. Susskind, Phys. Rev. **D62** (2000) 066004.

[6] J. Lukierski, P.C. Stichel and W.J. Zakrzewski, Ann. Phys. **260** (1997) 224.

[7] J. Gamboa, M. Loewe and J.C. Rojas, hep-th/0010220.

[8] D. Gross and N. Nekrasov, hep-th/0005204, JHEP 0007 (2000) 034.